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On (2 + 1)-dimensional Ermakov systems

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Abstract. Ermakov-type systems in 2 + 1 dimensions are introduced. Multiwave solutions of a (2+1)-dimensional Pinney equation and a modulated (2+1)-dimensional sine–Gordon equation are thereby constructed.

1. Introduction

The analysis of the coupled nonlinear ordinary differential equations known as Ermakov systems originated in 1880 [1]. There has since been an extensive literature devoted to their study [2–27]. The main theoretical interest in such systems centres around the fact that they admit a constant of motion, the Lewis–Ray–Reid (LRR) invariant, whereby a nonlinear superposition principle may be constructed. In terms of applications, Ermakov systems arise, in particular, in nonlinear elasticity [28, 29] and nonlinear optics [30–34].

In a recent development, it was shown that, remarkably, the LRR invariant is the key to a linearization procedure for the standard Ermakov system [24]. This linearization has been exploited by Athorne [27] to analyse the stability and periodicity of the particular Ermakov system derived in the two-layer shallow water context in [23].

It is natural to seek generalizations of Ermakov systems which preserve their attractive properties. In this connection, Athorne [25] has recently introduced a class of nonlinear dynamical systems which include as special cases the autonomous Ermakov system and Kepler-type central force problems with angular dependence on the force. It was shown that such nonlinear Kepler–Ermakov (KE) systems are linearizable via essentially the same procedure as that given in [24]. This represents a modification of the Whittaker transformation which allows the general problem of motion under central force to be reduced to the problem of motion in a parallel field of force [36]. Whittaker employed a constant of motion, namely the angular momentum, in his linearization procedure. In a similar manner, the linearization of the standard Ermakov system depends on the use of the LRR invariant. A generalized invariant and accompanying linearization was presented for the KE system in [25].

Here, by contrast, our concern is not with the construction of invariants for extended Ermakov systems. Rather, we present multiwave solutions to (2+1)-dimensional Ermakov systems. The procedure is a development of that adopted in [37–40] in connection with soliton-like solutions of nonlinear Klein–Gordon equations. In this paper, a broad class of multiwave solutions are presented which are appropriate, in particular, both for a (2 + 1)-dimensional Pinney equation and a (2 + 1)-dimensional modulated sine–Gordon

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equation. It is remarked that single-wave solutions of a modulated sine–Gordon equation were constructed by Ray via a Ermakov system in [17]. This work was later extended by Saermark [19] to produce kink solutions.

2. (2 + 1)-dimensional Ermakov systems

Here, we consider (2 + 1)-dimensional Ermakov-type systems

$$\begin{aligned} &\Box\phi + \omega^2\phi - \rho^{-3}H(\phi/\rho) = 0 \\ &\Box\rho + \omega^2\rho - \phi^{-3}J(\rho/\phi) = 0 \end{aligned} \tag{2.1}$$

where $\Box = g^{-1/2}\partial_\alpha g^{1/2}g^{\alpha\beta}\partial_\beta$ and ω may, in general, depend in an arbitrary manner on ϕ, ρ and their partial derivatives; $g_{\alpha\beta}$ is the metric of the underlying three-dimensional Lorentz spacetime for which the signature 1 is adopted.

A dimensional reduction of the system (2.1) is sought whereby it becomes the classical Ermakov system. This is effected here by requiring that ϕ and ρ depend only on two functions ξ and η which satisfy the conditions

$$\Box\xi = \Box\eta = 0 \qquad (\nabla\xi)^2 = 1 \qquad (\nabla\eta)^2 = 0 \qquad \nabla\xi\nabla\eta = 0. \tag{2.2}$$

The existence of such functions restricts the structure of the underlying spacetime to be of the form of a *pp*-wave, i.e. flat-space + tensor product of a null-vector with itself. It follows from (2.2) that $\nabla\eta$ is a geodesic null vector field and $\nabla\xi$ is space-like. Choosing ξ, η and σ, the affine parameter along the $\nabla\eta$ geodesics, as coordinates, we can write the metric as

$$ds^2 = d\eta(d\sigma + m(\xi, \eta, \sigma)\, d\eta) + d\xi^2. \tag{2.3}$$

For functions independent of σ, the d'Alembertian $\Box$ reduces to $\partial_{\xi\xi}|_\eta$ and (2.1) reduces to the Ermakov system

$$\begin{aligned} &\phi_{\xi\xi}|_\eta + \omega^2\phi = \rho^{-3}H(\phi/\rho) \\ &\rho_{\xi\xi}|_\eta + \omega^2\rho = \phi^{-3}J(\rho/\phi). \end{aligned} \tag{2.4}$$

It is important to note that the quantity η enters into the system as a parameter.

Now, (2.4) yields

$$\partial_\xi(\rho\phi_\xi - \phi\rho_\xi) = \rho^{-2}H(\phi/\rho) - \phi^{-2}J(\rho/\phi) \tag{2.5}$$

whence

$$r_{\tau\tau} = H(r) - r^{-2}J(r^{-1}) \tag{2.6}$$

where

$$r = \phi/\rho \tag{2.7}$$

$$\tau = \int \rho^{-2}|_\eta\, d\xi + \alpha(\eta) \tag{2.8}$$

and $\alpha(\eta)$ is arbitrary.

At this stage, it is convenient to proceed in terms of the independent variables

$$g = e^{\xi+\eta} \tag{2.9}$$

$$h = e^{\xi-\eta} \tag{2.10}$$

instead of ξ and η. The conditions (2.2) then become

$$\begin{aligned} &\Box g = g \qquad \Box h = h \\ &(\nabla g)^2 = g^2 \qquad (\nabla h)^2 = h^2 \\ &(\nabla g)(\nabla h) = gh. \end{aligned} \tag{2.11}$$

Integration of (2.6) yields

$$r_\tau = \sqrt{2\left\{I(\eta) + \int [H(r) - r^{-2}J(r^{-1})]\,\mathrm{d}r\right\}} \tag{2.12}$$

whence

$$\frac{\rho^4}{2}\left[g\partial_g(\phi/\rho) + h\partial_h(\phi/\rho)\right]^2 - \int^{\phi/\rho}\left[H(r) - r^{-2}J(r^{-1})\right]\,\mathrm{d}r = I(g/h). \tag{2.13}$$

The latter relation corresponds to the LRR invariant of standard Ermakov theory.

Let us now turn to the conditions (2.11) on g and h and introduce the ansatz

$$g = \sum_{i=1}^{N} e^{\theta_i+\delta_i} \tag{2.14}$$

$$h = \sum_{i=1}^{M} e^{\theta_i+\Delta_i} \tag{2.15}$$

where

$$\theta_i = p_i x + q_i y + w_i t \tag{2.16}$$

and δ_i, Δ_i are arbitrary phase constants. Insertion of (2.14) and (2.15) into (2.11) produces the restrictions

$$p_i^2 + q_i^2 - w_i^2 = 1 \tag{2.17}$$

$$(p_i - p_j)^2 + (q_i - q_j)^2 - (w_i - w_j)^2 = 0. \tag{2.18}$$

A geometric interpretation of the conditions (2.17) and (2.18) is suggested. Thus, if the vector $\boldsymbol{k}_1 = (p_1, q_1, w_1)$ is fixed subject to the requirement of its being a space-like unit vector then the endpoints of all the other $\boldsymbol{k}_i = (p_i, q_i, w_i)$ have to be on the intersection of the light cone of the endpoint of $\boldsymbol{k}_1$ and the unit hyperboloid; this intersection consists of two lines. The light cone of the endpoint of $\boldsymbol{k}_2$, however, will have only one line in common with the one of k_1. Hence, the endpoints of all the k_i lie on the unit hyperboloid and the straight null line determined by k_1 and the null vector $k_1 - k_2$.

The condition (2.18) is satisfied if

$$(p_i - p_j, q_i - q_j, w_i - w_j) = \alpha_{ij}(\lambda, \mu, \nu) \tag{2.19}$$

where

$$\lambda^2 + \mu^2 = \nu^2 \tag{2.20}$$

and (2.17) requires that

$$\lambda p_j + \mu q_j - \nu w_j = 0 \qquad \alpha_{ij} \neq 0. \tag{2.21}$$

In the above, the α_{ij} are arbitrary constants.

Once λ, μ, ν are chosen in accordance with (2.20), if p_1 is chosen arbitrarily, then q_1 and w_1 are determined via the constraints

$$p_1^2 + q_1^2 - w_1^2 = 1 \tag{2.22}$$

$$\lambda p_1 + \mu q_1 - \nu w_1 = 0. \tag{2.23}$$

The remaining (p_i, q_i, w_i) are then given by

$$p_i = p_1 + \alpha_{i1}\lambda \qquad q_i = q_1 + \alpha_{i1}\mu \qquad w_i = w_1 + \alpha_{i1}\nu. \tag{2.24}$$

Accordingly, multiwave solutions of the conditions (2.11) are obtained in the form

$$g = \mathrm{e}^{p_1 x + q_1 y + w_1 t} \sum_{l=1}^{N} \mathrm{e}^{\alpha_{l1}(\lambda x + \mu y + \nu t) + \delta_l} \tag{2.25}$$

$$h = \mathrm{e}^{p_1 x + q_1 y + w_1 t} \sum_{l=1}^{M} \mathrm{e}^{\alpha_{l1}(\lambda x + \mu y + \nu t) + \Delta_l} \tag{2.26}$$

where λ, μ, ν and p_1, q_1, w_1 are subject to the constraints (2.20), (2.22) and (2.23).

3. The $(2+1)$-dimensional Pinney equation

Here, attention is turned to the particular Ermakov system

$$\Box\phi + \omega^2\phi - k\phi^{-3} = 0 \tag{3.1}$$

$$\Box\rho + \omega^2\rho = 0 \tag{3.2}$$

coupled through ω^2. Thus, (3.1)–(3.2) represents a $(2+1)$-dimensional Pinney system. It is noted that higher-dimensional Pinney equations arise in quantum mechanical systems [30].

In this case, the canonical Ermakov system is

$$\phi_{\xi\xi}|_\eta + \omega^2\phi - k\phi^{-3} = 0 \tag{3.3}$$

$$\rho_{\xi\xi}|_\eta + \omega^2\rho = 0. \tag{3.4}$$

The associated first integral (2.13) yields

$$\rho\phi_\xi - \phi\rho_\xi = \sqrt{2I(\eta) - k\,(\rho/\phi)^2}. \tag{3.5}$$

Let ρ_1, ρ_2 be two linearly independent solutions of (3.4). The relation (3.5) yields

$$\begin{aligned}\rho_1\phi_\xi - \phi\rho_{1\xi} &= \sqrt{2I_1(\eta) - k\,(\rho_1/\phi)^2}\\ \rho_2\phi_\xi - \phi\rho_{2\xi} &= \sqrt{2I_2(\eta) - k\,(\rho_2/\phi)^2}\end{aligned} \tag{3.6}$$

and elimination of ϕ_ξ gives

$$\phi W(\rho_1, \rho_2) = \rho_2\sqrt{2I_1(\eta) - k\,(\rho_1/\phi)^2} - \rho_1\sqrt{2I_2(\eta) - k\,(\rho_2/\phi)^2} \tag{3.7}$$

where $W(\rho_1, \rho_2)$ is the Wronskian of ρ_1, ρ_2. Solution of (3.7) for ϕ produces the nonlinear superposition

$$\phi = [\bar{\lambda}(\eta)\rho_1^2 + 2\bar{\mu}(\eta)\rho_1\rho_2 + \bar{\nu}(\eta)\rho_2^2]^{1/2} \tag{3.8}$$

where $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$ satisfy the relation

$$\bar{\lambda}\bar{\nu} - \bar{\mu}^2 = k/W(\rho_1, \rho_2). \tag{3.9}$$

If $\omega = \omega(\eta)$ and we take

$$\rho_1 = \sin\omega\xi \qquad \rho_2 = \cos\omega\xi$$

then (3.8) yields

$$\phi = [\alpha(\eta) + \beta(\eta)\sin(2\omega(\eta)\xi + \gamma(\eta))]^{1/2} \tag{3.10}$$

where

$$\alpha^2 - \beta^2 = k/\omega. \tag{3.11}$$

Thus, on use of (2.25)–(2.26), it is seen that the $(2 + 1)$-dimensional Pinney equation (3.1) admits the class of multiwave solutions given by the expression (3.10) with

$$\xi = p_1x + q_1y + w_1t + \tfrac{1}{2}\ln[\Phi(x, y, t;\delta)\Psi(x, y, t;\Delta)] \tag{3.12}$$

$$\eta = \tfrac{1}{2}\ln[\Phi(x, y, t;\delta)/\Psi(x, y, t;\Delta)] \tag{3.13}$$

where

$$\Phi(x, y, t;\delta) := \sum_{i=1}^{N} \mathrm{e}^{\alpha_{i1}(\lambda x + \mu y + \nu t) + \delta_i} \tag{3.14}$$

$$\Psi(x, y, t;\Delta) := \sum_{i=1}^{M} \mathrm{e}^{\alpha_{i1}(\lambda x + \mu y + \nu t) + \Delta_i}. \tag{3.15}$$

It is noted that, for constant ω and k, there exists a Lagrangian for Pinney's equation. In figure 1, the energy density E associated with the Lagrangian is plotted at fixed time t for a solution of (3.1) with $\omega = k = 1$. This solution corresponds to the simple choice

$$\alpha = \cosh\gamma \qquad \beta = \sinh\gamma \qquad \gamma = 2\mathrm{e}^{-\eta^2} \tag{3.16}$$

together with

$$g = \mathrm{e}^{x+y-t} \qquad h = \mathrm{e}^{x-y+t}$$

so that $\xi = x$ and $\eta = y - t$. In such a frame, the object moves in the positive y-direction

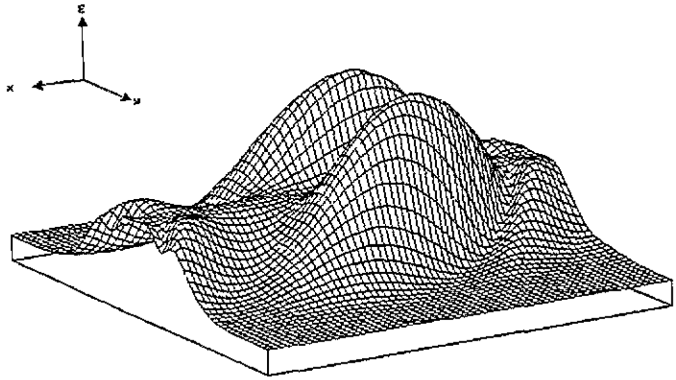

Figure 1. Energy density associated with the Lagrangian for (3.1) for the solution (3.16) at fixed t.

4. A $(2+1)$-dimensional modulated sine–Gordon equation

The $(2+1)$-dimensional Ermakov system

$$\Box\phi + \omega^2\phi - k\rho^{-3}\sin(\phi/\rho) = 0 \tag{4.1}$$

$$\Box\rho + \omega^2\rho = 0 \tag{4.2}$$

is considered next. This represents a $(2+1)$-dimensional sine–Gordon equation (4.1) modulated by a function $\rho(x, y, t)$ which, in turn, is governed by (4.2). It is noted that $(1+1)$-dimensional modulated systems of the type (4.1) and (4.2) have been considered by Ray [17] and Saermark [19]. Therein, travelling wave solutions were obtained.

In this case, $H = k\sin(\phi/\rho)$, $J = 0$ so that the canonical equation (2.6) becomes

$$r_{\tau\tau} = k\sin r. \tag{4.3}$$

Thus, if $r = r(\tau)$ is any solution of the nonlinear pendulum equation (4.3) then the nonlinear superposition

$$\phi = \rho(\xi) r\left\{\int \rho^{-2}|_\eta\, \mathrm{d}\xi + \alpha(\eta)\right\} \tag{4.4}$$

where

$$\rho_{\xi\xi}|_\eta + \omega^2\rho = 0 \tag{4.5}$$

and ξ, η are given by (3.12)–(3.15) provides a multiwave solution of the modulated $(2+1)$-dimensional sine–Gordon equation (4.1).

If, as in the $(1+1)$-dimensional case considered by Ray [17], the base solution

$$r = 4\tan^{-1}\exp[\tau/k^{1/2}] \tag{4.6}$$

of (4.3) is taken together with the particular solution

$$\rho = \sin\xi \tag{4.7}$$

of (4.5) with $\omega = 1$ then the nonlinear superposition (4.4) of (4.6) and (4.7) produces the multiwave solution

$$\phi = 4 \sin \xi \tan^{-1} \exp \frac{1}{k^{1/2}}(-\cot \xi + \alpha(\eta)) \tag{4.8}$$

of the modulated (2 + 1)-dimensional sine–Gordon equation

$$\Box\phi - \phi + k \sin^{-3} \xi \sin(\phi/ \sin \xi) = 0 \tag{4.9}$$

where ξ, η are given by (3.12)–(3.15).

In figure 2, the solution (4.8) is displayed at fixed time t in the special case $\xi = x, \eta = y - t, \alpha = 2e^{-\eta^2}$ and $k = 1$.

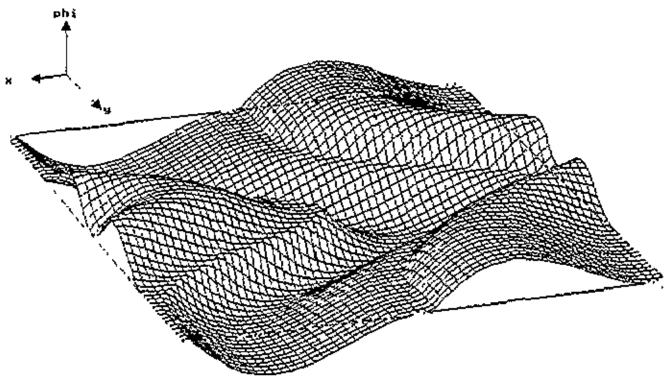

Figure 2. The solution (4.8) at fixed t.

Appendix. A (1+1)-dimensional Ermakov system. Application to nonlinear heat conduction

Here, a (1 + 1)-dimensional Ermakov system

$$-\alpha\phi^{k-1}\rho^{-k-3}\phi_t + \phi_{xx} + \omega(x)\phi = \rho^{-3}H(\phi/\rho) \tag{A1}$$

$$\rho_{xx} + \omega(x)\rho = \phi^{-3}J(\rho/\phi) \qquad \alpha \neq 0 \tag{A2}$$

is introduced.

Combination of (A1) and (A2) yields

$$-\alpha\phi^{k-1}\rho^{-k-2}\phi_t + \rho\phi_{xx} - \phi\rho_{xx} = \rho^{-2}H - \phi^{-2}J \tag{A3}$$

whence we obtain the canonical reduction

$$-\alpha r_{t'} + (r^{(1-k)/k}r_{x'})_{x'} = \Psi(r) \tag{A4}$$

where

$$x' = \int \rho^{-2}\,\mathrm{d}x \qquad t' = t \qquad r = \left(\frac{\phi}{\rho}\right)^k \tag{A5}$$

and $\Psi(r) := k(H(r^{1/k}) - r^{-2}J(r^{-1/k}))$.

Thus, it is seen that the (1+1)-dimensional Ermakov system (A1)–(A2) admits reduction to the nonlinear heat equation (A4) with a source term.

In particular, if $H = (\rho/\phi)^2 J$ then it is seen that the nonlinear modulated heat equation

$$-\alpha\phi^{k-1}\rho^{-k-3}\phi_t + \phi_{xx} - \rho_{xx}\rho^{-1}\phi = 0 \tag{A6}$$

admits the nonlinear superposition principle

$$\phi = \rho(x)r^{1/k}\left(\int \rho^{-2}\,\mathrm{d}x, t\right) \tag{A7}$$

where r is governed by the nonlinear heat equation

$$-\alpha r_{t'} + (r^{(1-k)/k}r_{x'})_{x'} = 0. \tag{A8}$$

The cases $k = -1, k = -3$ are of particular interest. Thus, if $k = -1$, (A8) is linearizable via a reciprocal transformation whereas if $k = -3$ it admits special group structure and associated similarity solutions.

References

[1] Ermakov V P 1880 *Univ. Isv. Kiev* **20** 1
[2] Lewis H R Jr 1967 *Phys. Rev. Lett.* **18** 510
[3] Lewis H R Jr 1968 *J. Math. Phys.* **9** 1976
[4] Lewis J R Jr and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[5] Khandekar D C and Lawande S V 1975 *J. Math. Phys.* **16** 384
[6] Ray J R and Reid J L 1979 *Phys. Lett.* **71A** 317
[7] Korsch H J 1979 *Phys. Lett.* **74A** 294
[8] Ray J R 1980 *Phys. Lett.* **78A** 4
[9] Wollenberg L S 1980 *Phys. Lett.* **79** 269
[10] Lutsky M 1980 *Phys. Lett.* **79A** 301
[11] Lutsky M 1980 *J. Math. Phys.* **21** 1370
[12] Ray J R and Reid J L 1981 *J. Math. Phys.* **22** 91
[13] Ray J R, Reid J L and Lutsky M 1981 *Phys. Lett.* **84A** 42
[14] Ray J R 1981 *Prog. Theor. Phys.* **65** 877
[15] Sarlet W and Ray J R 1981 *J. Math. Phys.* **22** 2504
[16] Sarlet W 1981 *Phys. Lett.* **82A** 161
[17] Ray J L 1981 *Lett. Nuovo Cimento* **30** 372
[18] Reid J L and Ray J R 1982 *J. Phys. A: Math. Gen.* **15** 2751
[19] Saermark K 1982 *Phys. Lett.* **90A** 5
[20] Reid J L and Ray J R 1982 *J. Math. Phys.* **23** 503
[21] Reid J L and Ray J R 1983 *J. Math. Phys.* **24** 2433
[22] Goedert J 1989 *Phys. Lett.* **136A** 391
[23] Rogers C 1989 *Research Report* Department of Mathematical Sciences, Loughborough University, UK
[24] Athorne C, Rogers C, Ramgulam U and Osbaldestin A 1990 *Phys. Lett.* **143A** 207
[25] Athorne C 1991 *J. Phys. A: Math. Gen.* **24** 945
[26] Leach P G L 1991 *Phys. Lett.* **158A** 102

[27] Athorne C J 1993 *Diff. Eqns.* to appear
[28] Shahinpoor M and Nowinski J L 1971 *Int. J. Nonlin. Mech.* **6** 193
[29] Rogers C and Ames W F 1989 *Nonlinear Boundary Value Problems in Science and Engineering* (New York: Academic)
[30] Lee R A 1984 *J. Phys. A: Math. Gen.* **17** 535
[31] Cervero J M and Lejareta J D 1990 *Quantum Opt.* **2** 333
[32] Goncharenko A M and Kukushkin 1990 *Vetsi Akad. BSSR. Ser. Fiz.* **6** 32
[33] Cervero J M and Lejareta J D 1991 *Phys. Lett.* **156A** 201
[34] Goncharenko A M, Logvin Yu A, Samson A M, Shapovatov P S and Turovets S I 1991 *Phys. Lett.* **160A** 138
[35] Athorne C 1991 *Preprint* No 91/29 Department of Mathematics, University of Glasgow
[36] Whittaker E T 1904 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press)
[37] Gibbon J D, Freeman N C and Davey A 1978 *J. Phys. A: Math. Gen.* **11** L93
[38] Gibbon J D, Freeman N C and Johnson R S 1978 *Phys. Lett.* **65A** 380
[39] Lou S and Guang G 1989 *J. Math. Phys.* **30** 1614
[40] Lou S and Chen W 1991 *Phys. Lett.* **156A** 260